Modeling Markets with Math: Intro to Theoretical Microeconomics

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1 Some multivariate calculus

1.1 Partial derivatives

Functions often take multiple variables. For example, $f(x, y)$ is a function of both x and y. If we let $z = f(x, y)$, then we can visualize f as a surface in xyz-space, where the height z_0 of a point (x_0, y_0, z_0) on that surface is equal to $f(x_0, y_0)$. The set of all points on the surface such that $z = k$, where k is a given constant, is called a *level set*. Graphically, a level set is a cross section of the curve parallel to the xy-plane. Lines in topographic maps are level sets.

How do we take the derivative of such a function, and what would that derivative mean? In single-variable calculus, the derivative of a function is the slope of its tangent line. However, with multiple variables, there are infinite tangent lines (in three dimensions, there is a tangent plane). One way to talk about derivatives when you have multiple variables is to take the derivative of each variable while holding the rest constant. Graphically, this means slicing the surface parallel to a certain axis (say, the x-axis) and taking the derivative with respect to x while holding the other variables fixed. This is called the *partial derivative* with respect to x, and it is denoted by $\frac{\partial f}{\partial x}$. (It is also denoted f_x and $\partial_x f$. The former is ambiguous and I advise against it, but it is used often.)

Example Let $f(x, y) = \sin xy + x^2 e^{3y}$. Then

$$\frac{\partial f}{\partial x} = y \cos xy + 2xe^{3y}$$
$$\frac{\partial f}{\partial y} = x \cos xy + 3x^2 e^{y}.$$

1.2 Optimization with Lagrange multipliers

Optimization in multiple variables is similar to optimization in a single variable. However, in economics, we are mostly concerned with optimization under a given constraint. This is expressed generally as optimizing $f(x, y)$ subject to the constraint $g(x, y) = c$, which is denoted

$$\max_{x,y} \{f(x, y)\}$$
$$\text{s.t. } g(x, y) = c,$$

if we are maximizing (and min if we are minimizing). The solution will be the point in $g(x, y) = c$ with the highest (or lowest) value for $f(x, y)$. Graphically, $g(x, y) = c$ is a curve in the xy-plane that can be projected onto the surface f. We want to walk along this curve and select the point at which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero. This will indicate to us that we are at a critical point. We will then assume that this critical point is a local extremum and, moreover, the global extremum we are looking for without checking the second-order derivative (economists are lazy).

This is computed with the method of Lagrange multipliers. Observe that at the optimal point, the curve $g(x, y) = c$ is tangent to a level set of f. (If these were not tangent, then the level set of f would cut through the curve $g(x, y) = c$, which would imply that f is increasing or decreasing along $g(x, y) = c$ at that point, i.e., the derivative is nonzero.) Therefore, the derivatives of $g(x, y) - c$ and $f(x, y)$ with respect to x and y will be parallel, i.e.,

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial}{\partial x}(g(x, y) - c) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial}{\partial y}(g(x, y) - c),$$

where λ is some constant (called the Lagrange multiplier). We can express this concisely by defining the Lagrangian,

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda(g(x, y) - c),$$

and optimizing $\mathcal{L}(x, y, \lambda)$ over x and y. Observe that taking the partial derivatives of the Lagrangian will give us the two equations above.

2 Utility theory

2.1 Preference relations

(Note that this section will not be very rigorous, because I am not assuming enough math to make it rigorous.)

Suppose there are two goods, called good-X and good-Y. A consumer can choose between different quantities of each good. Let the point (x, y) be a *consumption bundle* containing x units of good-X and y units of good-Y. The set of all such bundles is called her consumption set. We write $(x_1, y_1) \succsim (x_2, y_2)$ if the consumer thinks the bundle (x_1, y_1) is at least as good as the bundle (x_2, y_2). We call "$\succsim$" a preference relation; note that "$\succsim$" is similar to "$\geq$". We write $(x_1, y_1) \succ (x_2, y_2)$ if she strictly prefers the bundle (x_1, y_1) to (x_2, y_2), and we write $(x_1, y_1) \sim (x_2, y_2)$ if she is indifferent between the two. We say that her preferences are *rational* if they satisfy two conditions:

- for all bundles (x_1, y_1) and (x_2, y_2) in the consumption set, either $(x_1, y_1) \succsim (x_2, y_2)$ or $(x_1, y_1) \precsim (x_2, y_2)$ (this is called *completeness*), and
- for all bundles (x_1, y_1), (x_2, y_2), (x_3, y_3) in the consumption set, if both $(x_1, y_1) \succsim (x_2, y_2)$ and $(x_2, y_2) \succsim (x_3, y_3)$, then $(x_1, y_1) \succsim (x_3, y_3)$ (this is called *transitivity*).

These two conditions mean that the consumer has a preference (possibly indifference) between any two bundles, and her preferences aren't cyclical.

It will help us to define some properties about preference relations:

Definition. *A preference relation is* monotonic *if $x_1 > x_2$ and $y_1 > y_2$ implies that $(x_1, y_1) \succsim (x_2, y_2)$. (This means that at least as much of everything is at least as good.)*

Definition. *A preference relation is* locally nonsatiated *if given any bundle (x_1, y_1) in the consumption set, there exists a bundle (x_2, y_2) close enough to (x_1, y_1) such that $(x_2, y_2) \succ (x_1, y_1)$. (This means that one can always do a little bit better with a slight change in the bundle.)*

Definition. *A preference relation is* convex *implies that if two bundles are each preferable to a third, then any weighted average of the two bundles is also preferable to the third. Mathematically, if $(x_1, y_1) \succsim (x_3, y_3)$ and $(x_2, y_2) \succsim (x_1, y_1)$, then $t(x_1, y_1) + (1 - t)(x_2, y_2) \succsim (x_3, y_3)$ for all $0 < t < 1$.*

Note that convexity encapsulates the notion of diminishing marginal rates of substitution.

2.2 Utility functions

We want to define a function that expresses all these preferences as relations between numbers by mapping each bundle to a number. We call this a *utility function*. Given a consumption set and rational preferences over the consumption set, a utility function satisfies

$$u(x_1, y_1) \geq u(x_2, y_2) \text{ if and only if } (x_1, y_1) \succsim (x_2, y_2).$$

Notice that a utility function is a multivariable function, as discussed above. The function $u(x, y)$ maps a quantity of goods x and a quantity of goods y to a number, and all these numbers produce a surface in three-dimensional space.

It is very important to note that utility functions express ordinal, not cardinal, relations between bundles. That is to say, it indicates that a bundle is preferred to another bundle but not by how much. This is because a scaling of a utility function by any increasing function f is a valid utility function over that set of preferences. Mathematically, if

$$u(x_1, y_1) \geq u(x_2, y_2) \text{ if and only if } (x_1, y_1) \succsim (x_2, y_2).$$

then we also have

$$f(u(x_1, y_1)) \geq f(u(x_2, y_2)) \text{ if and only if } (x_1, y_1) \succsim (x_2, y_2).$$

Therefore, the utility of a bundle is *not* the happiness gained from that bundle or the usefulness of that bundle. It is merely a representation of how preferable it is compared to other bundles. This is a consequence of the choice of ordinal over cardinal utility. Economists choose ordinal utility over cardinal utility because it is incredibly difficult (if not impossible) to measure something like happiness with a yardstick that can be used for everyone.

A level set for a utility function is called an *indifference curve*, because the consumer is indifferent between all bundles in that level set.

Note that it isn't obvious that a utility function even exists for a given preference relation. However, it can be proven that if a preference relation is rational, then a utility function exists that represenents that preference relation. If the preference relations is also continuous and strictly monotonic, then the utility function is continuous.

2.3 Consumer choice

We assume that a rational consumer will always choose the most preferred bundle from a set of affordable bundles. We call this set the budget set. given an income m and prices p_x, p_y of good-X and good-Y, respectively, we define the budget set as all bundles (x, y) such that $p_x x + p_y y \leq m$. We model this rational behavior by maximizing the consumer's utility function over the budget set. Mathematically, we wish to solve the following optimization problem:

$$\max_{x,y} \{u(x, y)\}$$
$$\text{s.t. } p_x x + p_y y \leq m.$$

If the preference relation is locally nonsatiated, we can change the inequality in the constraint (called the budget constraint) to equality:

$$\max_{x,y} \{u(x, y)\}$$
$$\text{s.t. } p_x x + p_y y = m.$$

This maximization will yield two functions $x(p_x, p_y, m), y(p_x, p_y, m)$ called the demand functions, which indicate how much of each good the consumer will choose given their prices and her income.

Example Utility functions are often Cobb-Douglas functions: $u(x, y) = x^\alpha y^\beta$. Let's solve this explicit optimization problem:

$$\max_{x,y} \left\{x^\alpha y^\beta\right\}$$
$$\text{s.t. } p_x x + p_y y = m$$
$$\mathcal{L} = x^\alpha y^\beta + \lambda(p_x x + p_y y - m)$$
$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^\beta + \lambda p_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^\alpha y^{\beta-1} + \lambda p_y = 0$$

$$
\begin{aligned}
\frac{\alpha y}{\beta x} &= \frac{p_x}{p_y} \Rightarrow y = \frac{p_x}{p_y}\frac{\beta}{\alpha}x \\
m &= p_x x + p_y \frac{p_x}{p_y}\frac{\beta}{\alpha}x = \left(1 + \frac{\beta}{\alpha}\right) p_x x \\
x &= \frac{m}{\left(1+\frac{\beta}{\alpha}\right) p_x} \\
y &= \frac{p_x}{p_y}\frac{\beta}{\alpha}\frac{m}{\left(1+\frac{\beta}{\alpha}\right) p_x} = \frac{m}{\left(1+\frac{\alpha}{\beta}\right) p_y}
\end{aligned}
$$

Example Instead of giving the consumer an income, let's give her an endowment ω_x of good-X and an endowment of ω_y of good-Y. She can immediately consume her endowment or sell some or all of her endowment to buy other goods. Her budget constraint is now $p_x x + p_y y = p_x \omega_x + p_y \omega_y$. Let's also assume a slightly different utility function, where $\beta = 1 - \alpha$ and $\alpha < 1$.

$$
\begin{aligned}
&\max_{x,y} \left\{x^\alpha y^{1-\alpha}\right\} \\
&\text{s.t. } p_x x + p_y y = p_x \omega_x + p_y \omega_y \\
&\mathcal{L} = x^\alpha y^{1-\alpha} + \lambda(p_x x + p_y y - p_x \omega_x + p_y \omega_y) \\
&\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} + \lambda p_x = 0 \\
&\frac{\partial \mathcal{L}}{\partial y} = (1-\alpha) x^{\alpha} y^{-\alpha} + \lambda p_y = 0 \\
&\frac{\alpha y}{(1-\alpha)x} = \frac{p_x}{p_y} \Rightarrow y = \frac{p_x}{p_y}\frac{1-\alpha}{\alpha}x \\
&p_x x + p_y \frac{p_x}{p_y}\frac{1-\alpha}{\alpha}x = p_x \omega_x + p_y \omega_y \\
&x = \alpha \left(\omega_x + \frac{p_y}{p_x}\omega_y\right) \\
&y = (1-\alpha) \left(\omega_x + \frac{p_y}{p_x}\omega_y\right)
\end{aligned}
$$

2.4 Shortcomings of utility theory

Here are some possible shortcomings of utility theory:

- Preferences might not be transitive.
- Consumers probably don't maximize their utility.
- Utility is ordinal, not cardinal.
- Maybe it is more important to maximize something other than utility (see Amartya Sen's capability approach or Rawls theory of justice).

What else can you think of?

3 Exchange

Suppose now that we have multiple consumers, each with a certain endowment of goods who are free to trade among each other, taking prices as given. We will now denote consumer i's utility as u^i, her demand for good x as x^i, and her endowment of good x as ω_x^i. Note that the superscripts are not exponents; they are just labels. (The notation gets messy.) We will need a quick definition:

Definition. *An allocation of goods across consumers is* feasible *if the total goods allocated is at most the total goods available in the market.*

3.1 Competitive equilibrium

We define a *competitive equilibrium* (also called Walrasian equilibrium) to be the set of prices and allocations that maximizes each consumer's individual utility (i.e., allows them to maximize their own individual utility through exchange) and is feasible (i.e., there are no leftover goods or supply meets demand). That is to say, if consumers individually act in their own interest, a (feasible) equilibrium will emerge. We will mathematically formalize this definition for the case of two consumers A, B and two kinds of goods x, y:

Definition. *A* competitive equilibrium *is the set of prices p_x, p_y and allocations x^A, y^A, x^B, y^B that solves the following maximization problems,*

$$\max_{x^A,y^A} \left\{u^A(x^A, y^A)\right\} \qquad \max_{x^B,y^B} \left\{u^B(x^B, y^B)\right\}$$
$$\textit{s.t. } p_x x^A + p_y y^A = p_x\omega_x^A + p_y\omega_y^A \qquad \textit{s.t. } p_x x^B + p_y y^B = p_x\omega_x^B + p_y\omega_y^B,$$

and satisfies the following feasibility constraints, i.e.,

$$x^A + x^B = \omega_x^A + \omega_x^B$$
$$y^A + y^B = \omega_y^A + \omega_y^B.$$

Note that it is not guaranteed that a competitive equilibrium exists unless every consumer's utility function is continuous, strictly convex, and strictly monotonic. These are nontrivial assumptions.

Example Let's solve a competitive equilibrium for the set of prices. Suppose $u^A(x^A, y^A) = (x^A)^\alpha(y^A)^{1-\alpha}$ and $u^B(x^B, y^B) = (x^B)^\beta(y^B)^{1-\beta}$. Then we need to solve the following optimization problems for each consumer's demand functions:

$$\max_{x^A,y^A} \left\{(x^A)^\alpha(y^A)^{1-\alpha}\right\} \qquad \max_{x^B,y^B} \left\{(x^B)^\alpha(y^B)^{1-\alpha}\right\}$$
$$\text{s.t. } p_x x^A + p_y y^A = p_x\omega_x^A + p_y\omega_y^A \qquad \text{s.t. } p_x x^B + p_y y^B = p_x\omega_x^B + p_y\omega_y^B.$$

As we saw in the previous example, the solutions to the above optimizations are the following demand functions:

$$x^A = \alpha\left(\omega_x^A + \frac{p_y}{p_x}\omega_y^A\right) \qquad y^A = (1-\alpha)\left(\frac{p_x}{p_y}\omega_x^A + \omega_y^A\right)$$
$$x^B = \beta\left(\omega_x^B + \frac{p_y}{p_x}\omega_y^B\right) \qquad y^B = (1-\beta)\left(\frac{p_x}{p_y}\omega_x^B + \omega_y^B\right).$$

We also have the feasibility constraints:

$$x^A + x^B = \omega_x^A + \omega_x^B$$
$$y^A + y^B = \omega_y^A + \omega_y^B.$$

We now have six variables $x^A, y^A, x^B, y^B, p_x, p_y$ and six equations. We can solve for the relative price $\frac{p_y}{p_x}$:

$$\begin{aligned}
x^A + x^B &= \omega_x^A + \omega_x^B \\
\alpha\left(\omega_x^A + \frac{p_y}{p_x}\omega_y^A\right) + \beta\left(\omega_x^B + \frac{p_y}{p_x}\omega_y^B\right) &= \omega_x^A + \omega_x^B \\
\frac{p_y}{p_x}\left(\alpha\omega_y^A + \beta\omega_y^B\right) + \left(\alpha\omega_x^A + \beta\omega_x^B\right) &= \omega_x^A + \omega_x^B \\
\frac{p_y}{p_x} &= \frac{(1-\alpha)\omega_x^A + (1-\beta)\omega_x^B}{\alpha\omega_y^A + \beta\omega_y^B}.
\end{aligned}$$

3.2 The First Welfare Theorem

We want to define some normative concepts to qualify equilibriums.

Definition. *A feasible allocation of goods across consumers is* Pareto efficient *or* Pareto optimal *if there is no other feasible allocation that is weakly preferred ("$\succsim$") by all consumers and strictly preferred ("$\succ$") by some consumer.*

Note that Pareto optimality is not concerned with distribution or inequality. E.g., an allocation in which all goods are given to one consumer with strictly monotonic preferences is Pareto efficient.

Definition (First Welfare Theorem)**.** *If preferences are locally nonsatiated, then a competitive equilibrium is Pareto efficient.*

Proof. Suppose not. Let $(x_1^A, y_1^A), (x_1^B, y_1^B), p_x, p_y$ be the competitive equilibrium. Then there exists a feasible allocation $(x_2^A, y_2^A), (x_2^B, y_2^B)$ such that $(x_2^A, y_2^A) \succ (x_1^A, y_1^A)$ and $(x_2^B, y_2^B) \succsim (x_1^B, y_1^B)$ (without loss of generality). Observe that the second allocation must be more expensive than the first allocation for consumer A; otherwise, she would have chosen the second allocation when maximizing her utility. Similarly, the second allocation must be at least as expensive as the first allocation for consumer B. Therefore,

$$p_x x_2^A + p_y y_2^A > p_x x_2^A + p_y y_2^A$$
$$p_x x_2^B + p_y y_2^B \geq p_x x_2^B + p_y y_2^B$$
$$p_x x_2^A + p_y y_2^A + p_x x_2^B + p_y y_2^B > p_x x_2^A + p_y y_2^A + p_x x_2^B + p_y y_2^B$$

Combining the above with the feasibility and budget constraints, we have

$$p_x(x_2^A + x_2^B) + p_y(y_2^A + p_y y_2^B) > p_x \omega_x^A + p_y \omega_x^A + p_x \omega_x^B + p_y \omega_y^B.$$

Contradiction. □

It is interesting and helpful to consider why local nonsatiation is necessary. Also, note that the First Welfare Theorem is silent about the desirability of the distribution.

3.3 The Second Welfare Theorem

Somewhat of a converse can also be proven.

Theorem (Second Welfare Theorem)**.** *Given a Pareto efficient allocation in which all consumers have a positive endowment of all goods and in which preferences are convex, continuous, locally nonsatiated, and monotonic, there exists a set of prices such that the allocation and prices form a competitive equilibrium.*

(This is tedious to prove and requires more math than we have assumed.)

This theorem implies that any Pareto efficient allocation can implemented through competitive markets. Note that a benevelont social planner would have to know a nearly impossible amount of information about every consumer's preferences and endowments to identify a Pareto efficient allocation. Furthermore, the planner would need immense power to enforce the wealth transfers necessary for the allocation such that citizens don't avoid the wealth transfers. Thus, the Second Welfare Theorem gives many economists significant reasons to believe in the power of free markets, which yield Pareto efficient allocations by the First Welfare Theorem. Remember, however, that Pareto efficient is not a very strong or even that desirable quality of an economic system.